

Interaction of two one-dimensional Bose-Einstein solitons: Chaos and energy exchange

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We analyze the bright soliton interactions of a Bose-Einstein condensate in quasi-one-dimensional traps putting an emphasis on integrability break down due to a trapping potential. In particular, we derive a simple analytical model, which describes well all major features of soliton dynamics including chaos and energy exchange between interacting solitons induced by the trapping potential.

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I. INTRODUCTION

Bose-Einstein condensation (BEC) was first observed in 1995 in a series of remarkable experiments on vapors of rubidium [1] and sodium [2]. Later BEC of lithium [3] and hydrogen [4] was also demonstrated. These discoveries initiated an avalanche of further works dealing both with experimental and theoretical aspects of BEC (see the recent review [5]). Soon after the first experimental observations of the BEC phenomenon it was realized that coherent dynamics of the condensate wave function may lead to the formation of bright [6], dark [7], and vortex [8] *solitons*—self-localized formations of increased (bright) or decreased (dark or vortex) wave function density.

However, theoretical studies of soliton dynamics in BECs were usually limited to investigation of single soliton evolution under the influence of external fields and perturbations (see, e.g., Ref. [6]) or to analysis of weakly nonlinear (low BEC density) case where generalized versions of the standard linear coupled-mode theory hold (see, e.g., [9,10]). We would like to go far beyond this analysis and present a consistent analytical approach for studying long-term evolution of interacting solitons in dense (strongly nonlinear) condensates. In this paper we start with the simplest case of one-dimensional bright BEC solitons, developing a full-scale analytic model and comparing its predictions with direct numerical modeling. In particular, we discover the existence of stable two-soliton bound states (bisolitons) and observe dynamical chaos caused solely by soliton interactions in a stationary trap.

The rest of the paper is organized as follows: in Sec. II we start with the Gross-Pitaevskii equation and derive a soliton interaction ordinary differential equation (ODE) system; in Sec. III we investigate the static properties and stability of stationary bisoliton states; in Sec. IV we suggest the modified ODE system to describe the long-term dynamics of interacting (colliding) solitons; in Sec. V this model is used to quantify the energy exchange between the colliding solitons and subsequent development of chaos; finally, Sec. VI contains conclusions and discussion.

II. BASIC MODEL

We consider the macroscopic dynamics of BEC in a strongly anisotropic trapping potential $\tilde{V} = \tilde{V}(x, y/\mu, z/\mu)$,

where $\mu \ll 1$ is the anisotropy parameter. Such a potential forms a cigar-shaped trap oriented along the x direction. In this case the collective dynamics may be described by the one-dimensional Gross-Pitaevskii (GP) equation. Details of the derivation and normalization are very similar to that found in Ref. [6]. The (1+1)-dimensional macroscopic dynamics of the condensate wave function is governed by the equation:

$$i \frac{\partial \psi}{\partial t} + \frac{\partial^2 \psi}{\partial x^2} + \lambda |\psi|^2 \psi - V(x) \psi = 0, \quad (1)$$

where the nonlinearity parameter λ is defined as $\lambda = -2Na a_l/a_0^2$. In this definition N is the number of particles trapped in a BEC state, a is the ground state scattering length, a_0 and a_l are condensate sizes (without self-interaction) in transverse (y, z) and longitudinal (x) directions, respectively. [The transverse directions y and z have been integrated out in obtaining the GP equation (1).] The parameter λ is positive for condensates with self-attraction (negative scattering length; this regime was observed for BEC in lithium [12]). The specific form of trapping potential $V(x)$ depends on the details of the experimental setup.

We note that for validity of the quasi-one-dimensional approximation (1), the condition $a_0 \sqrt{a_l} / \sqrt{8\pi N} |a| > a_0$ (i.e., healing length is larger than the transverse size of the trap) should be satisfied. Taking this requirement into account and choosing sample parameter values as $a = -1.45$ nm (as for ⁷Li) and $a_0 = 1.0$ μ m, we estimate the physical range of our nonlinear parameter as $\lambda > 7.0 \times 10^{-5} N^2$. We note that, in principle, there are two ways to reduce the effective value of λ to relatively small values $|\lambda| \sim 1$: (i) using a magnetic-field-induced Feshbach resonance technique of Ref. [13] to reduce the scattering length parameter a or (ii) simply working with very low BEC densities (small number N of trapped atoms in a BEC state). However for typical BEC experiments ($a \sim 1.0$ nm, $N > 10^3$) the nonlinear parameter is large ($\lambda \sim 10^2 - 10^4$).

Although Eq. (1) is not integrable analytically, it possesses two important integrals of motion, which we will refer to as *energy*:

$$E = \int_{-\infty}^{\infty} \left\{ \left| \frac{\partial \psi}{\partial x} \right|^2 + V(x) |\psi|^2 - \frac{1}{2} \lambda |\psi|^4 \right\} dx, \quad (2)$$

and *mass*:

$$Q = \int_{-\infty}^{\infty} |\psi|^2 dx. \quad (3)$$

Equation (1) without an external potential [i.e., for $V(x) \equiv 0$] is an integrable model [14] and its one-soliton solutions may be presented in the form,

$$\psi(x, t) = \tilde{\psi}(x - Ct, t) e^{i\beta t}, \quad (4)$$

where C and β are soliton velocity and energy level, respectively, and complex *stationary*, i.e., t independent, one-soliton solutions $\tilde{\psi}$ are given by

$$\tilde{\psi}(\beta, C, \tilde{x}) = \frac{\sqrt{2\tilde{\beta}}}{\sqrt{\lambda}} \operatorname{sech}(\sqrt{\tilde{\beta}\tilde{x}}) e^{iC\tilde{x}/2}, \quad (5)$$

where $\tilde{\beta} \equiv \beta - C^2/4$, $\tilde{x} \equiv x - Ct$. If $\lambda \gg 1$, solitons given by Eqs. (5) can be used as a initial conditions in numerical analysis of soliton collisions and as zero-order approximations for construction of asymptotic interaction theory. In the analysis below, we choose $\tilde{\beta} = \lambda^2/16$ effectively normalizing the mass invariant (3) to unity for a single soliton (4). Note, that the total mass value may be different from $Q = 1$ if we have more than one single soliton, e.g., it takes the value $Q = 2$ for a bisoliton analyzed in the next section.

Using the methods of Ref. [15] we can derive a general system of ODEs for the soliton parameters, describing the adiabatic interaction of two almost identical BEC solitons in an external trap. This is based on two major assumptions that (i) the relative distance r between solitons is large in comparison to their size ($r \gg 1/\lambda$, the approximation of well-separated solitons) and (ii) relative soliton velocity $\Delta C \equiv C_2 - C_1$ is small ($\Delta C \ll 1$). Below we give an outline of the derivation procedure omitting the technical details (for a similar detailed derivation see, e.g., Ref. [16]). An alternative approach to the description of soliton interaction was used in Ref. [17].

We take two well-separated one-soliton solutions of Eq. (1) as the zeroth approximation of a nonstationary two-soliton solution. In other words we look for a solution of Eq. (1) in the form

$$\begin{aligned} \psi \equiv \psi_1 + \psi_2 = & \tilde{\psi}_1(x - x_1, T) \exp(i\phi_1) \\ & + \tilde{\psi}_2(x - x_2, T) \exp(i\phi_2), \end{aligned} \quad (6)$$

where subscripts 1 and 2 refer to first and second soliton respectively, $T \equiv \varepsilon t$ ($\varepsilon \ll 1$), and ϕ_i and x_i are soliton phases and center positions given by

$$\begin{aligned} \phi_i = & \int_0^T \beta_i(T') dT' + \phi_i^{(0)}, \\ x_i = & \int_0^T C_i(T') dT' + x_i^{(0)}, \end{aligned} \quad (7)$$

where $i = 1, 2$, and $\phi_i^{(0)}$ and $x_i^{(0)}$ are constants that define the initial soliton positions and phases. Note that we allow all internal parameters of both solitons (i.e., β_1 , β_2 , C_1 , and C_2) to depend on a *slow* variable T . Then we look for an asymptotic two-soliton solution of Eq. (1) in the form of an infinite series with ε being a small parameter of the asymptotic procedure. This approach is self-consistent only if certain compatibility conditions are satisfied. These compatibility conditions lead to a system of ordinary differential equations that is much simpler than the original model (1), but is still too complex to provide immediate physical insight. However, this system can be simplified further using additional assumptions.

Let us consider *two almost identical solitons*, which also have opposite initial velocities. In other words, initially $C_1 = -C_2$. Using these assumptions we can obtain the following analytical system for adiabatically changing soliton parameters:

$$\begin{aligned} -2 \frac{\partial Q_1}{\partial \beta_1} \dot{\phi}_1 + \frac{\partial U}{\partial \phi_1} &= 0, \\ -2 \frac{\partial Q_2}{\partial \beta_2} \dot{\phi}_2 + \frac{\partial U}{\partial \phi_2} &= 0, \\ Q_1 \ddot{x}_1 + \frac{\partial U}{\partial x_1} + 2Q_1 \frac{dV(x_1)}{dx_1} &= 0, \\ Q_2 \ddot{x}_2 + \frac{\partial U}{\partial x_2} + 2Q_2 \frac{dV(x_2)}{dx_2} &= 0, \end{aligned} \quad (8)$$

where $\phi_j(t)$ and $x_j(t)$ denote soliton phases and center positions and the potential U can be written in terms of soliton overlap integrals as

$$U = -2\lambda \cos \phi \operatorname{Re} \int_{-\infty}^{\infty} (|\tilde{\psi}_1|^2 \tilde{\psi}_1 \tilde{\psi}_2^* + |\tilde{\psi}_2|^2 \tilde{\psi}_2 \tilde{\psi}_1^*) dx, \quad (9)$$

where $\phi \equiv \phi_2 - \phi_1$ is the phase difference between two solitons. In the following we will use the normalization conditions $Q_1 = Q_2 \equiv 1$ ($\beta_1 = \beta_2 = \lambda^2/16$). Then the potential U may be approximated as

$$U(\phi, r) = -\lambda^2 \cos \phi e^{-\lambda|r|/4}, \quad (10)$$

where $r \equiv x_2 - x_1$.

System (8) may be simplified further by the standard exchange of absolute phase variables ϕ_i for a relative phase variable ϕ :

$$\begin{aligned} M_\phi \ddot{\phi} + \frac{\partial U}{\partial \phi} &= 0, \\ M_x \ddot{x}_1 + \frac{\partial U}{\partial x_1} + 2M_x \frac{dV}{dx_1} &= 0, \end{aligned} \quad (11)$$

$$M_x \ddot{x}_2 + \frac{\partial U}{\partial x_2} + 2M_x \frac{dV}{dx_2} = 0,$$

where the ‘‘phase mass’’ M_ϕ is defined by the equation

$$-2M_\phi^{-1} = \left(\frac{\partial Q_1}{\partial \beta_1} \right)^{-1} + \left(\frac{\partial Q_2}{\partial \beta_2} \right)^{-1}. \quad (12)$$

Taking into account the normalization we have $M_x = Q_1 = Q_2 = 1$ and $M_\phi = -8/\lambda^2$.

III. STATIONARY BISOLITON

First, we investigate the stationary case considering two solitons with equal amplitudes (and masses). As follows from system (11) a stationary solution requires the following conditions to be satisfied

$$\frac{\partial U}{\partial \phi} = 0,$$

$$\frac{dV}{dx_1} + \frac{dV}{dx_2} = 0, \quad (13)$$

$$\frac{dV}{dx_1} - \frac{dV}{dx_2} = \frac{\partial U}{\partial r}.$$

From the first of these equations and Eq. (10) it is easy to find the stationary values of phase difference $\phi_0 = 0, \pi$. The interaction of solitons is attractive for $\phi_0 = 0$ and repulsive for $\phi_0 = \pi$. System (13) has no solutions for $V(x) \equiv 0$; therefore two interacting solitons cannot form a stationary solution of the GP equation in free space [18]. However, in the presence of a confining potential the balance of external forces and of mutual repulsion can lead to the existence of a stationary two-soliton (bisoliton) solution of the GP equation (1).

Let us suppose that stationary distance between the centers of solitons $r_0 \ll D$, where D is the characteristic length of the potential $V(x)$. Near the minimum of the potential $V_{\min} = V(x_0)$

$$\frac{dV}{dx_1} \approx -\frac{hr}{2}, \quad \frac{dV}{dx_2} \approx \frac{hr}{2}, \quad (14)$$

where $h = d^2V/dx^2$ at $x = x_0$. Then the second equation in the system (13) is automatically satisfied and the third equation yields

$$hr = -\frac{\partial U}{\partial r}. \quad (15)$$

The linearization of system (11) about the stationary solution $\phi = \phi_0$, $x_1 = -r_0/2$, and $x_2 = r_0/2$ yields

$$\Delta \ddot{\phi} = -\Omega^2 \Delta \phi, \quad \Delta \ddot{x}_i = -\omega^2 \Delta x_i, \quad (16)$$

where $\Delta \phi = \phi - \phi_0$, $\Delta x_i = x_i - x_0$, $\Omega^2 = \lambda^2 U(\phi_0, r_0)/8$, $\omega^2 = 2h$, and $i = 1, 2$. Taking into account Eq. (10) we can see

that for $\phi = \pi$ and for a potential with a minimum ($h > 0$) bisoliton state is always stable. Quantitative details of stationary bisoliton structure may be also obtained using the system (11). To demonstrate the effectiveness of our analytic tools we make detailed calculations for the bisoliton family trapped due to the harmonic potential $V(x) = x^2/2$. For such a potential the value of r_0 is given by

$$R = \frac{\lambda^4 \sinh R}{32 \cosh^2 R}, \quad (17)$$

where $R = \lambda r_0/4$.

It is straightforward to show that Eqs. (11) can be presented in a standard Hamiltonian form with an explicit expression for the Hamiltonian (energy) function:

$$H = p_1^2 + p_2^2 - \frac{\lambda^2}{8} p_\phi^2 + \frac{1}{2} U(\phi, r) + V(x_1) + V(x_2). \quad (18)$$

The energy of the stationary state is given by the expression

$$E = \frac{r_0^2}{4} + \frac{\lambda^2}{4 \cosh R}. \quad (19)$$

It should be noted that the energy functional of two solitons in free space at $r_0 \rightarrow \infty$ tends to a nonzero value, namely,

$$E_0(\lambda) = -\frac{\lambda^2}{24}. \quad (20)$$

The properties of the stationary bisoliton solution may be also analyzed by a variational approach. We choose the trial function in the form of the direct sum of two free-space solitons with π relative phase shift:

$$\Phi(x, \lambda; a) = C \left(\frac{1}{\cosh z_+} - \frac{1}{\cosh z_-} \right), \quad (21)$$

where a is a variational parameter, corresponding to half separation between the centers of solitons, $C^2 = \lambda \sinh R / (8(\sinh R - R))$ is the normalization constant, $R = a\lambda/2$, and $z_\pm = \lambda(x \pm a)/4$. After the substitution of this trial function Eq. (2) and integration over x , it is easy to find the energy functional $E_v(a)$. The condition of extremum $dE_v/da = 0$ yields the transcendental equation for variational parameter a_{ex} . Then the value of $E_v(a_{ex})$ can be also obtained.

We used both analytical approaches to find the dependencies $R(\lambda)$ and $E(\lambda)$. These dependencies were also compared with the direct numerical results.

It can be seen from Figs. 1 and 2 that for $\lambda \gg 1$ the agreement between the analytical and numerical calculations is excellent, which supports the validity of both analytical approaches. In addition, we confirmed dynamical stability of stationary bisolitons by direct modeling of the GP equation (1) for few types of the external trapping potential $V(x)$. It appears that bisolitons may be stable with respect to strong perturbations.

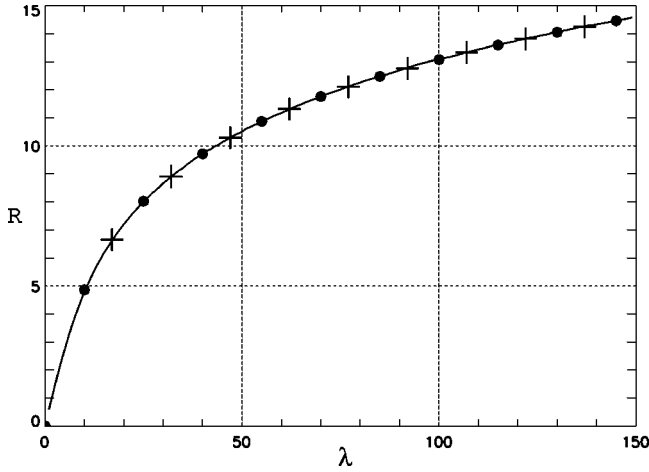


FIG. 1. Dependence of scaled intersoliton distance R vs nonlinearity parameter λ . Solid line corresponds to the direct numerical results, crosses—to the prediction of VA, filled circles denote the roots of Eq. (17).

Finally we would like to note that we also investigated the stability of *dynamical* (quasiperiodical) two-soliton structure, which, in contrast to *stationary* bisoliton considered above, depends on both x and t . This nonstationary soliton solution is well known for the GP equation with $V(x) \equiv 0$ (see, e.g., Refs. [20,21]). For this integrable case such a solution is *periodic* in t and may be expressed as a complex fractional structure of hyperbolic and trigonometric functions. Any perturbation, e.g., higher-order nonlinear corrections (see, e.g., Ref. [22] for an example) or nonzero $V(x)$ will break the integrability of free-space GP equation and make the two-soliton structure quasiperiodic due to emission of small amplitude radiation. Note, however, that some perturbations lead to quasistable evolution regime (slow decay of quasiperiodic soliton towards a single stationary soliton as in Ref. [22]), whereas other types of perturbations may lead to fast instability development and split into two nonequal solitons [23]. Our numerical analysis has shown that in all analyzed

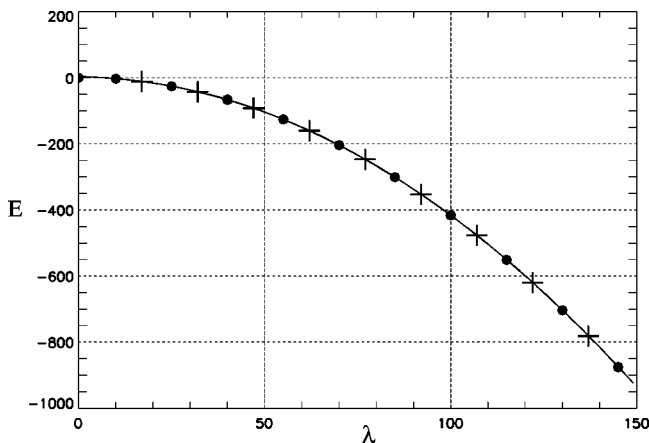


FIG. 2. Dependence of the bisoliton energy E vs nonlinearity parameter λ . Solid line corresponds to the direct numerical results, crosses—to the prediction of VA $E_v(a_{ex})$, filled circles—to the results of ODE model approximation $E + E_0$ [see Eqs. (19),(20)].

cases with nonzero potential $V(x)$ the later scenario takes place. Thus, the quasiperiodical nonstationary bisoliton proves to be an unstable formation and decays into two unbound interacting solitons.

IV. REDUCED MODEL

Now we pass on to the study of dynamical soliton interactions in an external potential field. For studying this case we shall transform the system of equations (8) into a phenomenological model that will serve our purposes far beyond the limits of validity of the original equations.

It is well known that although the model of the potential interaction (10) is derived from the assumption of well-separated and slowly moving solitons, it is applicable to the description of soliton collisions in free space, where it describes correctly the spatial displacements and the phase shift of colliding—that is, spatially overlapping at some time—solitons [15]. Still the system (8) is unacceptable for description of the colliding solitons in the presence of the external potential $V(x) \sim 1$. Although during the collision of solitons with $\lambda \gg 1$ their interaction prevails over the external forces, the alterations of motion of the soliton centers induced by the external potential, albeit small, eventually turn into large corrections to values of phases and finally violate the conservation of the energy of the system.

There are two noticeable ways to bypass this difficulty. The first one is to improve the derivation of the system of ODE to include terms of higher orders in λ^{-1} . Although the way to proceed to higher orders of the perturbation theory is known [19], it does not promise to be effective since the main problem is the extrapolation of the equations that are based on the supposition of weak interaction of solitons to the domain of strong interaction. We use an alternative approach that is described below.

The analysis of system (11) shows that for the collision of the solitons in the free space with relative velocities $v \sim 1$ almost at any values of the initial phase ϕ_0 apart from some small interval $|\phi_0| \leq \lambda^{-2}$, the phase difference ϕ at the moment of the closest approach of the solitons reaches the value $\phi = \text{sgn}(\phi_0)\pi$. Thus nearly for any initial conditions the interaction of colliding solitons eventually has the character of strong repulsion.

Therefore it is possible to neglect the evolution of the phase difference altogether, to ascribe the phase the constant value $\phi = \pi$, and to reduce the model to the system with two degrees of freedom with the equations of motion

$$\begin{aligned} \ddot{x}_1 + \frac{\partial U_\pi}{\partial x_1} + 2 \frac{dV}{dx_1} &= 0, \\ \ddot{x}_2 + \frac{\partial U_\pi}{\partial x_2} + 2 \frac{dV}{dx_2} &= 0, \end{aligned} \quad (22)$$

where

$$U_\pi(r) = U(\pi, r) = \lambda^2 e^{-\lambda|r|/4}. \quad (23)$$

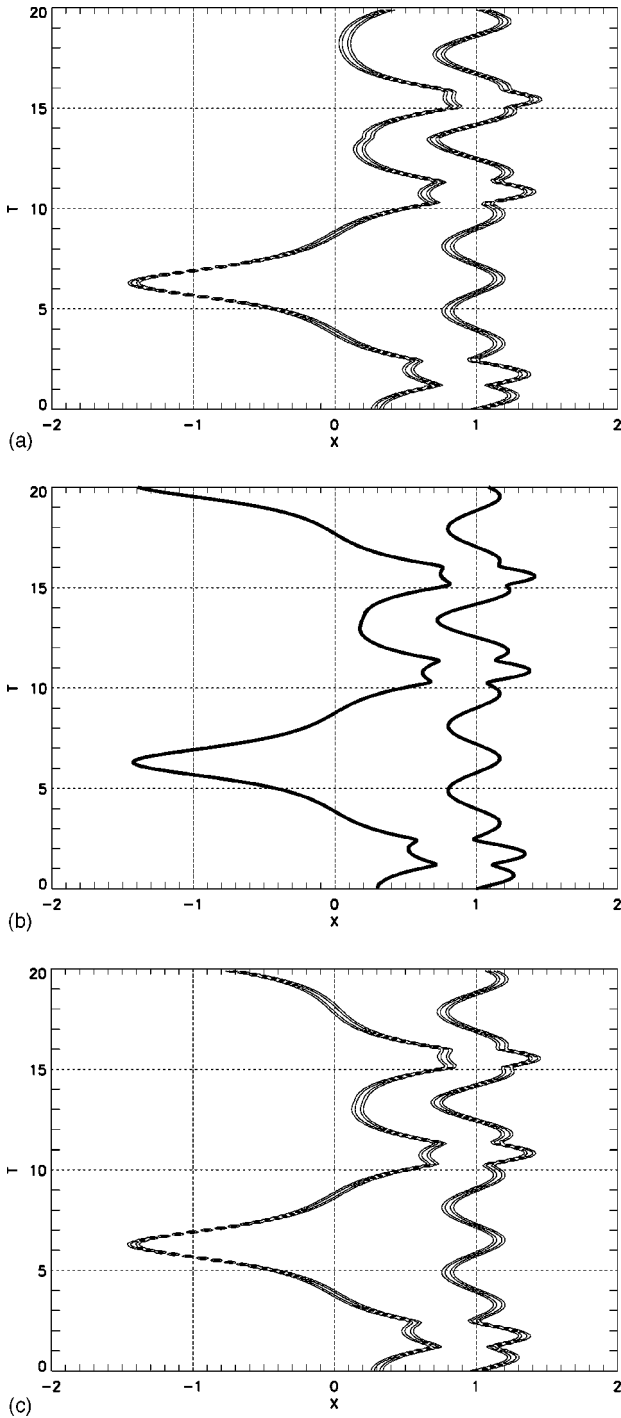


FIG. 3. Dynamics of two interacting solitons within a double well potential. Top and bottom figures are contour plots in soliton amplitude showing the numerical solution of the PDE (1) for the standard set of parameters (see text) with $\phi = \pi/2$ and $\phi = \pi$, respectively. The middle figure is the solution of the reduced (phase independent) ODE system (22) with the same initial conditions. The bold lines represent the trajectory of the soliton centers.

In Fig. 3 we plot the motion of two solitons of the system with the nonlinearity parameter $\lambda = 100$ and the initial conditions $x_1 = 0.3$, $x_2 = 1.0$, $\dot{x}_1 = 0.0$, $\dot{x}_2 = 0.6$ in the external strongly nonlinear potential of the double well

$$V(x) = -\frac{x^2}{2} + \frac{x^4}{4}. \quad (24)$$

In what follows this case will be called standard set of parameters. We note in passing that the properties of solutions of the nonlinear Schrodinger equation (NLS) in the external double-well potential for moderate values of the nonlinearity parameter $\lambda \sim 1$, that permit us to use the approximation of coupled modes, recently became the object of intensive studies for stationary [9] as well as for nonstationary [10,11] cases.

The top and bottom graphs in Fig. 3 were obtained by direct numerical solution of the Eq. (1), where the initial conditions ($t=0$) were taken in the form of the direct sum of two (separated) free-space single soliton solutions (4). The corresponding parts of Fig. 3 are contour plots of the soliton density $|\psi|^2$; the boundary contours correspond to 40% of the maximum density. In solving this equation numerically we have employed the standard split step Fourier or beam propagation method (BPM). By using a grid with 512 points, a transverse step size $\delta x \approx 7.8 \times 10^{-3}$, and a propagation step size $\delta t = 5 \times 10^{-6}$ the method conserves the mass Q to 10^{-6} accuracy. Choosing the mass of the exact one soliton to be normalized to one fixes the internal soliton parameter $\beta = \lambda^2/16$. As we wish to consider the case of $\lambda \gg 1$ and therefore large β , a small δt was chosen to contend with the fast phase rotation a large β causes in the modeling. The middle graph plotted is the numerical solution of the system (22).

The comparison of graphs shows that the discrepancies between the evolution of centers of solitons become noticeable only after the elapse of considerable amount of time. In our example, one of the solitons from the pair with $\phi = \pi$ crosses the symmetry line of the potential at $t = 18$, whereas in the pair with $\phi = \pi/2$ it reflects from the central hill of the potential at this moment. However, for the purposes we are going to use our system (22) the exact detailed form of the trajectory is not important; furthermore, in a sense it is inaccessible for reasons that are discussed in the following paragraph.

V. CHAOS AND ENERGY EXCHANGE

The apparent irregularity of motion, which is observable in Fig. 3, suggests its chaotic character. The reduced system (22) belongs to the class of Hamiltonian autonomous systems with two degrees of freedom, with Cartesian coordinates x_1 and x_2 and appropriate canonically conjugate momenta, that describes motion in some static potential $W(x_1, x_2)$. Although a member of this class, the famous Henon-Heiles model [24], served as the foundation of one of the paradigms of modern chaotic dynamics, very little attention (if any) has been paid to systems with a potential of the form $V(x_1) + V(x_2) + U(|x_1 - x_2|)$ that we deal with here.

Chaotic motion of a system of two particles—infinitesimal rigid balls—in a uniform gravitational field above a rigid floor was studied in Ref. [25]. The range of the interaction potential in this model is exactly zero; that leads to essential differences from our case: in particular, the model becomes trivial in the case of equally massive par-

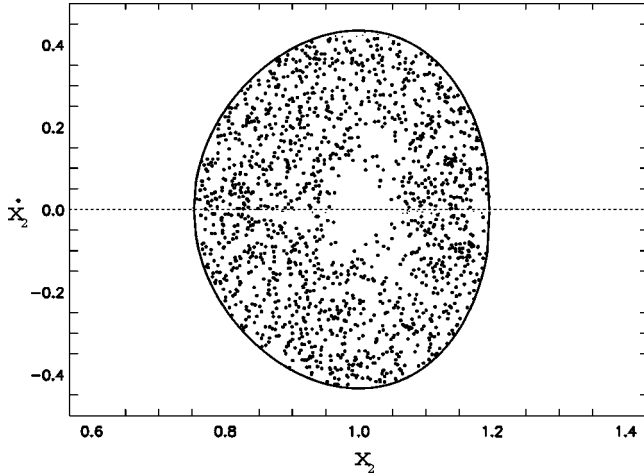


FIG. 4. Poincaré section of particle moving in a double well potential with position x_2 and velocity \dot{x}_2 at moments when the other particle is about to enter the right hand well (i.e., $x_1=0$ and $\dot{x}_1>0$).

ticles. Next, the chaotic motion of two interacting particles was extensively studied in the context of the problem of behavior of two particles in a Paul trap [26–28]. However, this system is usually treated by use of dissipative nonautonomous models, that differ qualitatively from our case.

To qualify the motion as chaotic, one needs to study the evolution of the system for very long times. In Fig. 4 we plotted the projection of the phase trajectory with standard initial conditions on the Poincaré section. The values of coordinate x_2 and velocity \dot{x}_2 of the right particle are taken at the moments when the left particle had the coordinate $x_1=0$ and positive velocity, $\dot{x}_1>0$. It is seen that the trajectory belongs to a chaotic component that covers most of the accessible phase space.

The Lyapunov exponent σ was calculated for this component and was found to be $\sigma=0.18\pm 0.01$.

In Fig. 5 we plotted the relative autocorrelation function of velocity $v=\dot{x}$ of one of the particles,

$$B_v(\tau) = \overline{v(t+\tau)v(t)[v^2(t)]^{-1}} \quad (25)$$

along with the dependence given by the interpolation formula

$$\tilde{B}_v(\tau) = (1 + \tau)^{-\nu} \cos(\omega\tau). \quad (26)$$

The agreement of numerical points with Eq. (26) strongly implies the power law decay of the correlations.

The system of two particles with repulsive interaction moving in a one-dimensional confining potential resembles the well-known Fermi accelerator—a particle moving in the one-dimensional box with an oscillating wall [29]. In our case each particle could be viewed as a sort of oscillating wall with respect to its neighbor. This analogy suggests more detailed study of the process of energy exchange.

Let's consider a general problem of one-dimensional finite motion of two particles of equal masses $m_1=m_2=1$ and

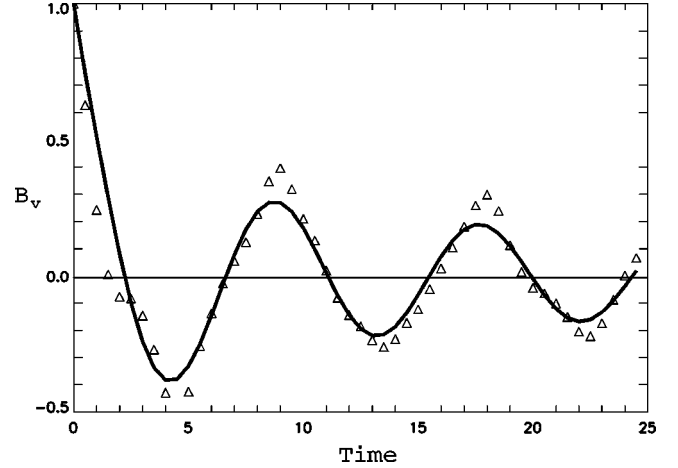


FIG. 5. Relative autocorrelation function of velocity $B_v(\tau)$. Triangles are the points calculated numerically from system (22). Solid line is given by the interpolation formula (26) with the best fit parameters $\nu=0.568$ and $\omega=0.710$.

coordinates x_1 and x_2 . The particles are subjected to an external potential $V(x_1)=V(x_2)$ and their interaction is described by the potential $U(|x_1-x_2|)$. The interaction potential is assumed to be rapidly (e.g., exponentially) vanishing for $|x_1-x_2|\geq d$ and the interaction range d is assumed to be small in comparison with the characteristic length D of motion in the external potential. Since $d\ll D$, the motion could be described as independent oscillations of two particles in the external potential that are, from time to time, interrupted by rapid collisions.

At first let's consider the collisions in the absence of the external field. The equation of motion for the distance between the particles $r=x_1-x_2$ is

$$\ddot{r} = -2 \frac{dU(r)}{dr}. \quad (27)$$

It has the first integral of energy $E_r = \dot{r}^2/2 + 2U(r)$.

A quantity that we need for the following is the difference of time that is spent by colliding particles in the range of the length $A\gg d$ in the absence of the interaction [$U(r)\equiv 0$] and in its presence. For the interaction rapidly vanishing at large r this difference has a finite limit for $A\rightarrow\infty$ that we shall call the *time shift* τ .

Two cases of collisions must be distinguished. In the first one the relative distance r changes its sign during the collision. This T case (T for “transmission”) corresponds to attractive [$U(r)<0$] or weak repulsive [$0<U(r)<E_r/2$] interactions. For the T case

$$\tau = \int_{-\infty}^{\infty} \left[\frac{1}{v} - \frac{1}{\sqrt{v^2 - 4U(r)}} \right] dr, \quad (28)$$

where $v = \sqrt{2E_r}$ is the magnitude of the relative velocity of infinitely separated particles. From Eq. (28) it is easy to see that $\tau>0$ for attractive and $\tau<0$ for weakly repulsive interactions.

In the second case the relative distance r retains its sign. This R case (R for “reflection”) corresponds to the strong repulsive interaction, when $E_r < \max 2U(r)$. For the R case

$$\tau = \int_{-\infty}^0 \frac{2}{v} dr - \int_{-\infty}^{r_*} \frac{2}{\sqrt{v^2 - 4U(r)}} dr, \quad (29)$$

where the turning point r_* is defined by the minimal root of the equation $v^2 - 4U(r_*) = 0$. For the exponential potential (10) the time shift is

$$\tau = \frac{16}{\lambda v} \ln \frac{\lambda}{v}. \quad (30)$$

It is positive for small v , changes its sign at $v = \lambda$ and remains negative to the limit of strong repulsion. This pattern of behavior is typical for any repulsive potential with a single extremum. We note in passing that for large λ and moderate $v \lesssim 1$ the time shift τ is much larger than the collision time θ , that for the exponential potential (10) is about $\theta \sim 16/\lambda v$.

Let's denote the (asymptotic) values of the partial energies of the particles $E_i(t) = \dot{x}_i^2/2$ before and after the collision as E_i^- and E_i^+ , respectively. In the absence of the external field in the T case the asymptotic values of energy of each particle is conserved,

$$E_1^+ - E_1^- = 0, \quad E_2^+ - E_2^- = 0, \quad (31)$$

whereas in the R case the particles exchange their energies,

$$E_2^+ - E_1^- = 0, \quad E_2^- - E_1^+ = 0. \quad (32)$$

The external potential violates these conservation laws. The description of energy exchange could be simply given with the assumption that the external potential V is slowly varying in comparison with the interaction potential U . Then in the first approximation the external forces acting on each particle might be assumed equal and described by a constant (uniform) force F at the “point” of collision:

$$-\frac{dV(x_1)}{dx_1} \simeq -\frac{dV(x_2)}{dx_2} \simeq F. \quad (33)$$

With this assumption the equations of motion for the interparticle distance r and for the coordinate of the center of mass $s = (x_1 + x_2)/2$ are independent,

$$\ddot{r} = -2 \frac{dU(r)}{dr}, \quad \ddot{s} = F, \quad (34)$$

and could be integrated. The partial energies of the particles are now defined as $E_i(t) = \dot{x}_i^2/2 - Fx_i$. The Eqs. (34) permit us to find the change of energy induced by the external potential. In the T case

$$E_1^+ - E_1^- = \Delta E = E_2^- - E_2^+, \quad (35)$$

and in the R case

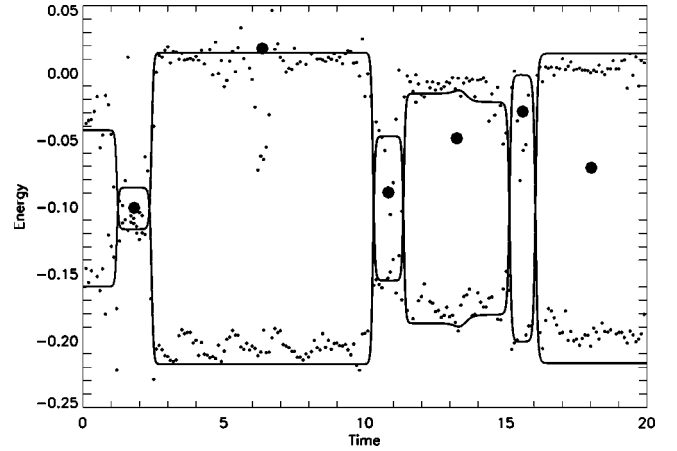


FIG. 6. Time dependence of the energy of solitons at the standard set of parameters (see text). Scattered small dots are taken from the numerical solution of the PDE (1), with $\phi = \pi/2$, solid lines—from the solution of the ODE system (22), large dots—from the model of rigid rods, Eq. (38).

$$E_1^- - E_2^+ = \Delta E = E_2^- - E_1^+, \quad (36)$$

where

$$\Delta E = \frac{\dot{r}}{2} F \tau. \quad (37)$$

Equation (37) states that as a result of the collision in the accelerating field ($F > 0$) with $\tau > 0$ the left particle (for the T case—the one that was on the left before the collision) receives an additional increment in energy.

The method that we described above is asymptotically exact in the limit of large λ ; however, it is insufficiently accurate for the standard set of parameters, that was chosen with regard to the possibility of numerical integration of the partial differential equation (PDE) (1). In this case the interaction range $d = 8\lambda^{-1} \ln(32/\lambda v) \approx 0.4$ is not small in comparison with the characteristic length of the potential $D \approx 1$. In five collisions out of six that could be seen in Fig. 3, at the moments of closest approach, external forces acting on the particles have different signs. That rules out the applicability of the approach based on assuming the validity of Eq. (33).

Another view on the energy exchange could be derived from the observation that the interaction potential $U_\pi(r)$ in Eq. (23) is rather steep, and the interacting particles could be replaced by rigid rods (one-dimensional balls) of radius d . The collision of rigid rods in the external potential leads to the exchange of kinetic energies of the particles. That yields

$$\Delta E = V(x_1) - V(x_2), \quad (38)$$

where values of x_i are taken at the moment of collision. Since for the collision of rigid rods $\tau = 2d/\dot{r}$, for the case of uniform field Eqs. (37) and (38) produce the same result $\Delta E = Fd$.

In Fig. 6 the values of partial energies of solitons calculated numerically from the GP equation using the BPM described earlier are plotted by scattered small points. The Gal-

ilean invariance of Eq. (1) gives “moving” solitons and the velocity \dot{x}_i of the i th soliton can be calculated as

$$\dot{x}_i = 2 \frac{\partial}{\partial x} \left\{ \arctan \left[\frac{\text{Im } \tilde{\psi}}{\text{Re } \tilde{\psi}} \right] \right\}, \quad (39)$$

at the center of the soliton x_i . Then the partial energies were calculated as $E_i(t) = \dot{x}_i^2/2 + 2V(x_i)$.

Solid lines show the time dependency of partial energies of the particles found from the system (22). Large points indicate values of the partial energy of the most energetic particle calculated from Eq. (38). The agreement of this estimate with numerical values is somewhat qualitative (the relative error in ΔE is about 30%), as expected. The model of rigid balls is based on the small value of the parameter ϵ that gives the ratio of average magnitude of the external forces $\langle |F| \rangle$ to the maximal force of interaction $\max |dU/dr|$. For the standard conditions $\langle |F| \rangle \approx 0.8$ and $\max |dU/dr| = \lambda v^2/16 \approx 6$, thus $\epsilon \sim 0.1$.

VI. DISCUSSION AND CONCLUSIONS

In this paper, starting from the substitution of two-soliton solution into Gross-Pitaevskii equation (1), we have found that if we are interested in the evolution of the spatial position of solitons in the external potential, the phenomenological model with two degrees of freedom (22) can be used with reasonably high accuracy. It is instructive that the more advanced system with three degrees of freedom (11), which works perfectly well for analysis of multisoliton stationary structures and their stability, becomes inadequate in describing the long-term soliton interaction dynamic. This somewhat paradoxical situation could be explained by the level of approximation: the equations of motion that we used have ignored the modification of the evolution of phases by the external potential. The derivation of equations that will be able to describe the evolution of soliton phases (as well as

coordinates) in the external field demands the development of the analytical model established in Sec. II to higher orders of the perturbation theory and remains a challenging problem. This derivation would remove the necessity for another simplifying assumption used to obtain the model (11), namely that masses of interacting solitons Q_i are invariant in the course of their evolution. In fact, the numerical solution of the PDE (1) shows that each collision of solitons leads to some exchange of masses between them, on the order of magnitude of $\Delta Q/Q \sim 10^{-3}$.

Importantly the study of the system (22) revealed a chaotic nature of motion. Although the model belongs to a well-studied class, it has rather specific features. The specifics of this system partly come from the presence of three characteristic time scales: duration of the particle collisions $\theta \sim 16/\lambda v$, time shift $\tau \sim 16/\lambda v \ln(\lambda/v)$ and a typical time between the collisions $T \sim 2\pi$, and, correspondingly, three characteristic length scales: soliton width $\delta \sim 4/\lambda$, range of interaction $d \approx 8\lambda^{-1} \ln(2\lambda/v)$, and characteristic width of the potential well $D \sim 1$. The theory of the energy exchange between the solitons colliding in the external field that was sketched in Sec. V is exact in the limit $\lambda \rightarrow \infty$ and can be used to study the final (long-term) distribution of the partial energies of solitons for fixed total value of the energy of the system. This work is currently in progress.

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